The Integration of Exponential Function 指数函数积分

$$\int e^x dx = e^x + C \qquad \int e^u du = e^u + C$$

Example Integrating Exponential Functions

(1) $\int e^{3x+1} dx$

$u = 3x + 1, du = 3dx$

$$\int e^{3x+1} dx = \frac{1}{3}\int e^{3x+1}(3)dx = \frac{1}{3}\int e^u du = \frac{1}{3}e^u + C = \frac{1}{3}e^{3x+1} + C$$

(2) $\int 5xe^{-x^2} dx$

$u = -x^2, du = -2xdx$

$$\int 5xe^{-x^2} dx = \frac{-5}{2}\int e^{-x^2}(-2xdx) = \frac{-5}{2}\int e^u du$$

$$= \frac{-5}{2}e^u + C = \frac{-5}{2}e^{-x^2} + C$$

(3) $\int \frac{e^{\frac{1}{x}}}{x^2} dx$

$u = 1/x, du = \frac{-1}{x^2} dx$

$$\int \frac{e^{\frac{1}{x}}}{x^2} dx = -\int e^u du = -e^u + C = -e^{\frac{1}{x}} + C$$

The Integration of logarithmic Function 对数函数积分

$$\int \frac{1}{x}dx = \ln|x| + C \qquad \int \frac{1}{u}du = \ln|u| + C$$

Because $du = u'dx$, the second formula can also be written as

$$\int \frac{u'}{u}dx = \ln|u| + C$$

Example Integrating the Quotient Functions

(1) $\int \frac{3x^2+1}{x^3+x}dx = \ln|x^3+x| + C$ $\quad u = x^3 + x$

(2) $\int \frac{\sec^2 x}{\tan x}dx = \ln|\tan x| + C$ $\quad u = \tan x$

(3) $\int \frac{x+1}{x^2+2x}dx = \frac{1}{2}\int \frac{2x+2}{x^2+2x}dx$ $\quad u = x^2 + 2x$

$= \frac{1}{2}\ln|x^2+2x| + C$

(4) $\int \frac{1}{3x+2}dx = \frac{1}{3}\int \frac{3}{x^2+2x}dx$ $\quad u = 3x + 2$

$= \frac{1}{3}\ln|3x+2| + C$

(5) $\int \frac{2x}{(x+1)^2}dx$ $\quad let\ u = x+1, du = dx\ and\ x = u - 1$

$\int \frac{2x}{(x+1)^2}dx = \int \frac{2(u-1)}{u^2}du$ $\quad u = x + 1$

$= 2\int \frac{(u-1)}{u^2}du = 2\int \frac{1}{u}du - 2\int \frac{1}{u^2}du = 2\ln|u| - 2\left(\frac{u^{-1}}{-1}\right) + C$

$= 2\ln|u| + \frac{2}{u} + C = 2\ln|x+1| + \frac{2}{x+1} + C$

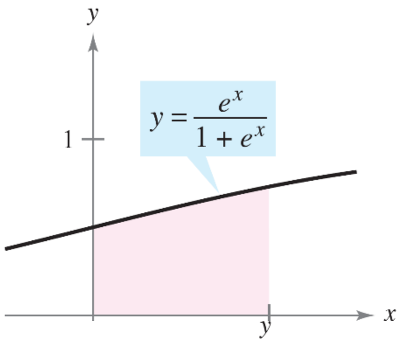
Example Finding Area Bounded by Exponential Functions

(1) $\int_0^1 \frac{e^x}{1+e^x}dx$

$u = 1 + e^x, u' = e^x$

$\int_0^1 \frac{e^x}{1+e^x}dx = \ln(1+e^x)\Big|_0^1$

$= \ln(1+e) - \ln 2$

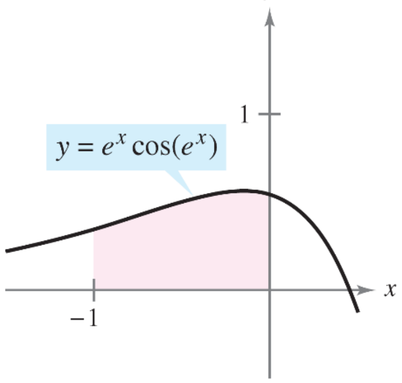


(2) $\int_0^1 [e^x \cos(e^x)]dx$

$u = e^x, du = e^x dx$

$\int_{-1}^0 [e^x \cos(e^x)]dx = \int_{-1}^0 \cos(e^x)\, de^x$

$= \sin e^x \Big|_{-1}^0 = \sin 1 - \sin(e^{-1})$



Integrals of Trigonometric Functions 三角函数积分

Example

$$\int \tan x\, dx = \int \frac{\sin x}{\cos x} dx = -\int \frac{-\sin x}{\cos x} dx \qquad u = \cos x \;\; u' = -\sin x$$

$$= -\int \frac{1}{\cos x} d(\cos x) = -\ln|\cos x| + C$$

Example

$$\int \sec x\, dx = \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x} dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx$$

Let $u = \sec x + \tan x \Rightarrow u' = \sec x \tan x + \sec^2 x$

$$\int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx = \int \frac{u'}{u} dx = \int \frac{1}{u} du \qquad u = \sec x + \tan x$$

$$= \ln|u| + C = \ln|\sec x + \tan x| + C$$

Integrals of the Six Basic Trigonometric Functions 基本三角函数积分

$$\int \sin x\, dx = -\cos x + c \qquad \int \cos x\, dx = \sin x + c$$

$$\int \tan x\, dx = -\ln|\cos x| + c \qquad \int \cot x\, dx = \ln|\sin x| + c$$

$$\int \sec x\, dx = \ln|\sec x + \tan x| + c \qquad \int \csc x\, dx = -\ln|\csc x + \cot x| + c$$

Exercise 1 (P360-99), find the indefinite integral.

99. $\int e^{5x}(5)dx$

$= \int e^{5x}d5x = e^{5x} + c$

100. $\int e^{-x^4}(-4x^3)dx$

$u = -x^4, du = -4x^3dx$

$\int e^{-x^4}(-4x^3)dx = \int e^u du = e^u + c = e^{-x^4} + \mathrm{c}$

101. $\int e^{2x-1}dx$

$u = 2x - 1, du = 2dx$

$\int e^{2x-1}dx = \frac{1}{2}\int e^{2x-1}(2dx) = \frac{1}{2}\int e^u du = \frac{1}{2}e^u + c = \frac{1}{2}e^{2x-1} + c$

102. $\int e^{1-3x}dx$

$u = 1 - 3x, du = -3dx$

$\int e^{1-3x}dx = \frac{-1}{3}\int e^{1-3x}(-3dx) = \frac{-1}{3}\int e^u du = \frac{-1}{3}e^u + c = \frac{-1}{3}e^{1-3x} + c$

103. $\int x^2e^{x^3}dx$

$u = x^3, du = 3x^2dx$

$\int x^2e^{x^3}dx = \frac{1}{3}\int e^{x^3}(3x^2dx) = \frac{1}{3}\int e^u du = \frac{1}{3}e^u + c = \frac{1}{3}e^{x^3} + c$

104. $\int e^x(e^x + 1)^2dx$

$u = e^x + 1, du = e^x dx$

$\int e^x(e^x + 1)^2dx = \int (u)^2du = \frac{u^3}{3} + c = \frac{1}{3}(e^x + 1)^3 + c$

105. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx$

$u = e^{\sqrt{x}}, du = e^{\sqrt{x}}\left(\frac{1}{2}x^{-\frac{1}{2}}\right)dx = \frac{e^{\sqrt{x}}}{2\sqrt{x}}dx$

$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx = 2\int \frac{e^{\sqrt{x}}}{2\sqrt{x}}dx = 2\int du = 2u + c = 2e^{\sqrt{x}} + \mathrm{c}$

106. $\int \frac{e^{1/x^2}}{x^3}dx$

$u = e^{1/x^2}, du = e^{1/x^2}(-2)(x^{-3})dx$

$\int \frac{e^{1/x^2}}{x^3}dx = \frac{-1}{2}\int \frac{e^{1/x^2}}{x^3}(-2)dx = \frac{-1}{2}\int du = \frac{-1}{2}u + c = \frac{-1}{2}e^{1/x^2} + c$

107. $\int \frac{e^{-x}}{1 + e^{-x}}dx$

$u = 1 + e^{-x}, du = e^{-x}(-dx)$

$\int \frac{e^{-x}}{1 + e^{-x}}dx = -\int \frac{-e^{-x}}{1 + e^{-x}}dx = -\int \frac{1}{u}du = -\ln|u| + c = -\ln|1 + e^{-x}| + c$

108. $\int \frac{e^{2x}}{1 + e^{2x}}dx$

$u = 1 + e^{2x}, du = e^{2x}(2dx)$

$\int \frac{e^{2x}}{1 + e^{2x}}dx = \frac{1}{2}\int \frac{e^{2x}}{1 + e^{2x}}(2dx) = \frac{1}{2}\int \frac{1}{u}du = \frac{1}{2}\ln|u| + c = \frac{1}{2}\ln|1 + e^{2x}| + c$

109. $\int e^x\sqrt{1 - e^x}dx$

$u = 1 - e^x, du = -e^x dx$

$\int e^x\sqrt{1 - e^x}dx = -\int \sqrt{u}du = -\frac{u^{\frac{3}{2}}}{3/2} + C = \frac{-2(1 - e^x)^{\frac{3}{2}}}{3} + C$

110. $\int \frac{e^x - e^{-x}}{e^x + e^{-x}}dx$

$u = e^x + e^{-x}, du = (e^x - e^{-x})dx$

$\int \frac{e^x - e^{-x}}{e^x + e^{-x}}dx = \int \frac{1}{u}du = \ln|u| + C = \ln(e^x + e^{-x}) + C$

111. $\int \frac{e^x + e^{-x}}{e^x - e^{-x}}dx$

$u = e^x - e^{-x}, du = (e^x + e^{-x})dx$

$\int \frac{e^x + e^{-x}}{e^x - e^{-x}}dx = \int \frac{1}{u}du = \ln|u| + C = \ln|e^x - e^{-x}| + C$

112. $\int \frac{2e^x - 2e^{-x}}{(e^x + e^{-x})^2} dx$

$u = e^x + e^{-x}, du = (e^x - e^{-x})dx$

$$\int \frac{2e^x - 2e^{-x}}{(e^x + e^{-x})^2} dx = \int \frac{2}{u^2} du = \frac{u^{-1}}{-1} + C = \frac{-1}{u} + C = \frac{-1}{e^x + e^{-x}} + C$$

113. $\int \frac{5 - e^x}{e^{2x}} dx$

$$\int \frac{5 - e^x}{e^{2x}} dx = \int \frac{5}{e^{2x}} dx - \int \frac{1}{e^x} dx = \int 5e^{-2x} dx - \int e^{-x} dx$$

$$= \frac{-5}{2} \int e^{-2x} d(-2x) + \int e^{-x} d(-x) = \frac{-5}{2} e^{-2x} + e^{-x} + C$$

114. $\int \frac{e^{2x} + 2e^x + 1}{e^x} dx$

$$\int \frac{e^{2x} + 2e^x + 1}{e^x} dx = \int e^x dx + 2 \int dx + \int e^{-x} dx = e^x + 2x - e^{-x} + C$$

115. $\int e^{-x} \tan(e^{-x})\, dx$

$u = e^{-x}, du = -e^{-x} dx$

$$\int e^{-x} \tan(e^{-x})\, dx = -\int \tan u\, du = \ln|\cos u| + C = \ln|\cos e^{-x}| + C$$

116. $\int \ln(e^{2x-1})\, dx$

$$\int \ln(e^{2x-1})\, dx = \int (2x - 1) dx = x^2 - x + C$$

Exercise 2 (P360-129), find the particular solution that satisfies the initial conditions.

129. $f''(x) = \frac{1}{2}(e^x + e^{-x}) \qquad f(0) = 1, f'(0) = 0$

$$f'(x) = \int \frac{1}{2}(e^x + e^{-x}) dx = \frac{1}{2}(e^x - e^{-x}) + C$$

$$f'(0) = \frac{1}{2}(e^0 - e^0) + C = 0 \Rightarrow C = 0$$

$$f'(x) = \frac{1}{2}(e^x - e^{-x})$$

$$f(x) = \int \frac{1}{2}(e^x - e^{-x}) dx = \frac{1}{2}(e^x + e^{-x}) + C$$

$$f(0) = \frac{1}{2}(e^x + e^{-x}) = 1 \Rightarrow C = 0$$

$$f(x) = \frac{1}{2}(e^x + e^{-x})$$

130. $f''(x) = \sin x + e^{2x} \qquad f(0) = \frac{1}{4}, f'(0) = \frac{1}{2}$

$$f'(x) = \int (\sin x + e^{2x}) dx = -\cos x + \frac{1}{2} e^{2x} + C$$

$$f'(0) = -1 + \frac{1}{2} + C = \frac{1}{2} \Rightarrow C = 1$$

$$f'(x) = -\cos x + \frac{1}{2} e^{2x} + 1$$

$$f(x) = \int \left(-\cos x + \frac{1}{2} e^{2x} + 1\right) dx = -\sin x + \frac{1}{2}\left(\frac{1}{2} e^{2x}\right) + x + C$$

$$f(0) = 0 + \frac{1}{4} + 0 + C = \frac{1}{4} \Rightarrow C = 0$$

$$f(x) = -\sin x + \frac{1}{4} e^{2x} + x$$

Exercise 3 (P340), find the indefinite integral.

11. $\int \frac{x^2 - 4}{x} dx$

$$= \int (x - \frac{4}{x})\, dx = \frac{x^2}{2} - 4\ln|x| + c$$

12. $\int \frac{x}{\sqrt{9 - x^2}} dx$

$u = 9 - x^2, du = -2dx$

$$\int \frac{x}{\sqrt{9 - x^2}} dx = \frac{-1}{2} \int \frac{-2x}{\sqrt{9 - x^2}} dx = \frac{-1}{2} \int \frac{1}{\sqrt{u}} du =$$

$$\frac{-1}{2}\int u^{-1/2}\,du = \left(\frac{-1}{2}\right)(2)u^{\frac{1}{2}} + c = -\sqrt{u} + c$$

13. $\displaystyle\int \frac{x^2+2x+3}{x^3+3x^2+9x}dx$

$u = x^3+3x^2+9x, du = (3x^2+6x+9)dx$

$$\int \frac{x^2+2x+3}{x^3+3x^2+9x}dx = \frac{1}{3}\int \frac{3x^2+6x+9}{x^3+3x^2+9x}dx$$

$$= \frac{1}{3}\int \frac{1}{u}du = \frac{1}{3}\ln|u| + c = \frac{1}{3}\ln|x^3+3x^2+9x| + c$$

14. $\displaystyle\int \frac{x(x+2)}{x^3+3x^2-4}dx$

$u = x^3+3x^2-4, du = (3x^2+6x)dx$

$$\int \frac{x(x+2)}{x^3+3x^2-4}dx = \int \frac{x^2+2x}{x^3+3x^2-4}dx = \frac{1}{3}\int \frac{3x^2+6x}{x^3+3x^2-4}dx$$

$$= \frac{1}{3}\int \frac{1}{u}du = \frac{1}{3}\ln|u| + c = \frac{1}{3}\ln|x^3+3x^2-4| + c$$

15. $\displaystyle\int \frac{x^2-3x+2}{x+1}dx$

$u = x+1, x = u-1, x^2-3x+2 = (u-1)^2 - 3(u-1) + 2$

$= \mathrm{u}^2 - 5u + 6$

$$\int \frac{x^2-3x+2}{x+1}dx = \int \frac{u^2-5u+6}{u}du = \int \left(u - 5 + \frac{6}{u}\right)du$$

$$= \frac{u^2}{2} - 5u + 6\ln|u| + c = \frac{(x+1)^2}{2} - 5(x+1) + 6\ln|x+1| + c =$$

16. $\displaystyle\int \frac{2x^2+7x-3}{x-2}dx$

$u = x-2, x = u+2, 2x^2+7x-3 = 2(u+2)^2 + 7(u+2) - 3$

$= 2\mathrm{u}^2 + 15u + 19$

$$\int \frac{2x^2+7x-3}{x-2}dx = \int \frac{2\mathrm{u}^2+15u+19}{u}du = \int 2u du + \int 15 du + \int \frac{19}{u}du$$

$$= 2(1/2)u^2 + 15u + 19\ln|u| + c = (x-2)^2 + 15(x-2) + 19\ln|x-2| + c$$

21. $\displaystyle\int \frac{(\ln x)^2}{x}dx$

$$= \int (\ln x)^2 d(\ln x) = \frac{(\ln x)^3}{3} + \mathrm{c}$$

22. $\displaystyle\int \frac{1}{x\ln x^3}dx$

$$= \int \frac{1}{x(3\ln x)}dx = \frac{1}{3}\int \frac{1}{\ln x}d(\ln x)$$

$$= \frac{1}{3}\int (\ln x)^{-1} d(\ln x) = \frac{1}{3}(\ln|\ln x|) + c$$

23. $\displaystyle\int \frac{1}{\sqrt{x+1}}dx$

$u = x+1, x = u-1$

$$\int \frac{1}{\sqrt{x+1}}dx = \int \frac{1}{\sqrt{u}}du = \int u^{-\frac{1}{2}}du$$

$$= \frac{u^{1/2}}{1/2} + c = 2\sqrt{u} + \mathrm{c} = 2\sqrt{x+1} + \mathrm{c}$$

24. $\displaystyle\int \frac{1}{x^{\frac{2}{3}}(1+x^{\frac{1}{3}})}dx$

$u = 1 + x^{\frac{1}{3}}, du = \frac{1}{3}x^{\frac{-2}{3}}dx$

$$3\int \frac{\frac{1}{3}x^{\frac{-2}{3}}}{(1+x^{\frac{1}{3}})}dx = 3\int \frac{1}{u}du = 3\ln|u| + c = 3\ln\left|1 + x^{\frac{1}{3}}\right| + c$$

Exercise 4 (P340), find the indefinite integral.

31. $\displaystyle\int \cot\frac{\theta}{3}d\theta$

$$= 3\int \cot\frac{\theta}{3}d\frac{\theta}{3} = 3\ln\left|\sin\frac{\theta}{3}\right| + c$$

32. $\displaystyle\int \tan 5\theta\, d\theta$

$$= \frac{1}{5}\int \tan 5\theta\, d(5\theta) = \frac{1}{5}(-\ln|\cos 5\theta|) + c = \frac{-1}{5}\ln|\cos 5\theta| + \mathrm{c}$$

33. $\int \csc 2x\,dx$

$= \frac{1}{2}\int \csc 2x\,d(2x) = \frac{1}{2}(-\ln|\csc 2x + \cot 2x|) + c = \frac{-1}{2}\ln|\csc 2x + \cot 2x| + c$

34. $\int \sec\frac{x}{2}dx$

$= 2\int \sec\frac{x}{2}d\frac{x}{2} = 2\ln\left|\sec\frac{x}{2} + \tan\frac{x}{2}\right| + c$

35. $\int (\cos 3\theta - 1)d\theta$

$= \int \cos 3\theta\,d\theta - \int d\theta = \frac{1}{3}\int \cos 3\theta\,d(3\theta) - \theta + c = \frac{1}{3}\sin 3\theta - \theta + c$

36. $\int (2 - \tan\frac{\theta}{4})d\theta$

$= 2\int d\theta - \int \tan\frac{\theta}{4}d\theta = 2\int d\theta - 4\int \tan\frac{\theta}{4}d\frac{\theta}{4}$

$= 2\theta - 4\left(-\ln\left|\cos\frac{\theta}{4}\right|\right) + c = 2\theta + 4\ln\left|\cos\frac{\theta}{4}\right| + c$

37. $\int \frac{\cos t}{1 + \sin t}dt$

$u = 1 + \sin t, du = \cos t$

$\int \frac{\cos t}{1 + \sin t}dt = \int \frac{1}{u}du = \ln|u| + c = \ln|1 + \sin t| + c$

38. $\int \frac{\csc^2 t}{\cot t}dt$

$= -\int \frac{1}{\cot t}d(\cot t) = -\ln|\cot t| + c$

39. $\int \frac{\sec x \tan x}{\sec x - 1}dx$

$u = \sec x - 1, du = \sec x \tan x$

$\int \frac{\sec x \tan x}{\sec x - 1}dx = \int \frac{1}{u}du = \ln|u| + c = \ln|\sec x - 1| + c$

40. $\int (\sec 2x + \tan 2x)dx$

$= \frac{1}{2}\int (\sec 2x + \tan 2x)d(2x) = \frac{1}{2}(\ln|\sec 2x + \tan 2x| - \ln|\cos 2x|) + c$

Exercise 5 (P340), solve the differential equation so that your solution can pass through the given point.

43. $\frac{dy}{dx} = \frac{3}{2 - x}$, $(1,0)$

$u = 2 - x, du = -dx$

$y = \int \frac{3}{2 - x}dx = (-3)\int \frac{-1}{2 - x}dx = (-3)\int \frac{1}{u}du$

$= -3\ln|u| + c = -3\ln|2 - x| + c$

$0 = -3\ln|2 - 1| + c \Rightarrow c = 3\ln|1| = 0$

$\Rightarrow y = -3\ln|2 - x|$

44. $\frac{dy}{dx} = \frac{2x}{x^2 - 9}$, $(0,4)$

$u = x^2 - 9, du = 2xdx$

$y = \int \frac{2x}{x^2 - 9}dx = \int \frac{1}{u}du = \ln|u| + c = \ln|x^2 - 9| + c$

$4 = \ln|9| + c \Rightarrow c = 4 - \ln 9$

$\Rightarrow y = \ln|x^2 - 9| + 4 - \ln 9$

45. $\frac{ds}{d\theta} = \tan 2\theta$, $(0,2)$

$s = \int \tan 2\theta\,d\theta = \frac{1}{2}\int \tan 2\theta\,d(2\theta) = -\frac{1}{2}\ln|\cos 2\theta| + c$

$2 = -\frac{1}{2}\ln|\cos 0| + c = -\frac{1}{2}\ln 1 + c = c$

$s = -\frac{1}{2}\ln|\cos 2\theta| + 2$

46. $\frac{dr}{dt} = \frac{\sec^2 t}{\tan t + 1}$, $(\pi, 4)$

$u = \tan t + 1, du = \sec^2 t$

$r = \int \frac{\sec^2 t}{\tan t + 1}dt = \int \frac{1}{u}du = \ln|u| + c = \ln|\tan t + 1| + c$

$4 = \ln|\tan \pi + 1| + c = \ln 1 + c \ \Rightarrow c = 4$

$r = \ln|\tan t + 1| + 4$

47. Determine the funciton f if $f''(x) = \dfrac{2}{x^2}$， $f(1) = 1, f'(1) = 1, x > 0$

$$f'(x) = \int \frac{2}{x^2} dx = \int 2x^{-2} dx = 2\frac{x^{-1}}{-1} + c = \frac{-2}{x} + c$$

$$f'(1) = 1 = \frac{-2}{1} + c \Rightarrow c = 1 + 2 = 3$$

$$f'(x) = \frac{-2}{x} + 3$$

$$f(x) = \int (\frac{-2}{x} + 3) dx = -2\ln x + 3x + c \quad (x > 0)$$

$f(1) = 1 = -2\ln 1 + 3 + c \Rightarrow c = 1 - 3 = -2$

$f(x) = -2\ln x + 3x - 2$

48. Determine the funciton f if $f''(x) = \dfrac{-4}{(x-1)^2} - 2, f(2) = 3, f'(2) = 0, x > 1$

$u = x - 1, du = dx$

$$f'(x) = \int \left[\frac{-4}{(x-1)^2} - 2\right] dx = \int \left[\frac{-4}{u^2} - 2\right] du = \int [-4(u^{-2}) - 2] du$$

$$= -4\frac{u^{-1}}{-1} - 2u + c = \frac{4}{u} - 2u + c = \frac{4}{x-1} - 2(x-1) + c$$

$$f'(2) = 0 = \frac{4}{2-1} - 2(2-1) + c = 4 - 2 + c \ \Rightarrow c = -2$$

$$f'(x) = \frac{4}{x-1} - 2(x-1) - 2$$

$$f(x) = \int \left[\frac{4}{x-1} - 2(x-1) - 2\right] dx$$

$u = x - 1, du = dx$

$$f(x) = \int \left[\frac{4}{x-1} - 2(x-1) - 2\right] dx = \int \left[\frac{4}{u} - 2u - 2\right] du$$

$$= 4\ln|u| - 2\frac{u^2}{2} - 2u + c = 4\ln(x-1) - (x-1)^2 - 2(x-1) + c \qquad (x > 1)$$

$f(2) = 3 = 4\ln(2-1) - (2-1)^2 - 2(2-1) + c = -1 - 2 + c \Rightarrow c = 6$

$f(x) = 4\ln(x-1) - (x-1)^2 - 2(x-1) + 6$

$= 4\ln(x-1) - x^2 + 7$

(此方法不好，将$f'(x)$化简为$\frac{4}{x-1} - 2x$比较不容易出错。)